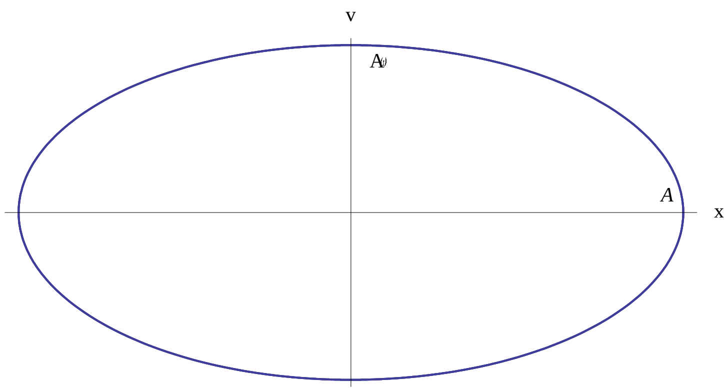
Teacher notes
Topic C

Simple harmonic oscillations

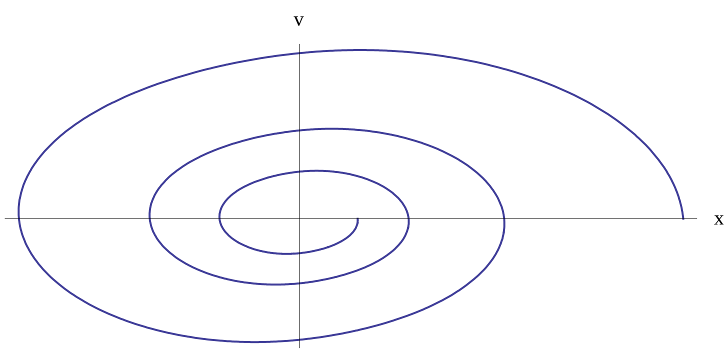
A basic equation in SHM is that relating speed to displacement: $v^2 = \omega^2(A^2 - x^2)$ which can also be rewritten as

$$\frac{v^2}{\omega^2 A^2} + \frac{x^2}{A^2} = 1$$

This is the equation of an ellipse if we plot *v* versus *x*:



The equation of the area *S* of this ellipse is $S = \pi(\omega A)A = \pi\omega A^2$ and so is proportional to the total energy of the motion, $E = \frac{1}{2}m\omega^2 A^2$. Thus, the area is $S = \frac{2\pi E}{m\omega}$. Hence, for damped motion where the total energy decreases the graph of *v* versus *x* looks like:



Question: In the SHM of a mass-spring system the mass is doubled while the amplitude stays the same. What happens t the total energy of the system?

The total energy is $E=\frac{1}{2}m\omega^2A^2$. It is tempting to state that the energy doubles, but we are ignoring the change in ω. For the mass-spring system $\omega^2=\frac{k}{m}$. Hence the energy does not change.

We could get this result faster by using instead $E=\frac{1}{2}kA^2$ from which it is immediately clear that the energy does not change.

The importance of SHM

SHM is important for two main reasons.

First, it is the basis for understanding waves. When a wave travels through a medium, the medium particles perform SHM with the same frequency as that of the wave.

The second reason is that for small amplitude, all oscillations are SHM. Consider a conservative force *F*: it is given by $F=-\frac{dU}{dx}$ where *U* is the potential energy function. Suppose *x* = *a* is a position of equilibrium. Taylor expanding *U* about *x* = *a*, we find

$$U(x)=U(a)+(x-a)\left.\frac{dU}{dx}\right|_{x=a}+\frac{1}{2}(x-a)^2\left.\frac{d^2U}{dx^2}\right|_{x=a}+\cdots$$

$$F=-\frac{dU(x)}{dx}=0-\left.\frac{dU}{dx}\right|_{x=a}-(x-a)\left.\frac{d^2U}{dx^2}\right|_{x=a}+\cdots$$

But $-\left.\frac{dU}{dx}\right|_{x=a}$ is the force at equilibrium, i.e. this is zero. Then

$$F=-(x-a)\left.\frac{d^2U}{dx^2}\right|_{x=a}+\cdots$$

If the oscillations have small amplitude, we may neglect the terms indicated by the dots and so

$$F\approx-(x-a)\left.\frac{d^2U}{dx^2}\right|_{x=a}=-k(x-a)$$

This says that the force is opposite to and proportional to the displacement $(x-a)$ and so SHM takes place. This becomes the standard SHM equation $a\approx-\omega^2\times\text{displacement}$ with $\omega^2=\frac{1}{m}\left.\frac{d^2U}{dx^2}\right|_{x=a}$.